Exercise 1.2.3

Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$.

Solution

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters the left side of a rod at $x = a$, then that same amount must exit the right side of it at $x = b$ for the temperature to remain the same. If more (less) thermal energy enters at $x = a$ than exits at $x = b$, then the amount of thermal energy in the rod will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of energy in} - \text{rate of energy out} = \text{rate of energy accumulation}$$

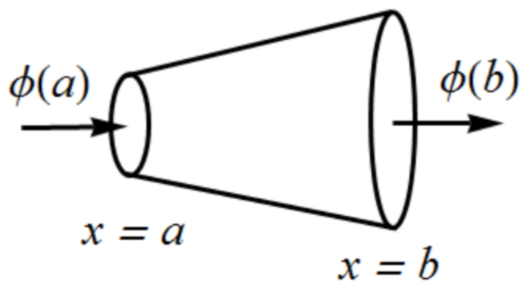

Figure 1: This is a schematic of the rod with variable cross-sectional area that the thermal energy flows through (integral formulation).

The flux is defined to be the rate that thermal energy flows through the rod per unit area, and we denote it by $\phi = \phi(x, t)$. If we let U represent the amount of energy in the rod, then the energy balance over it is

$$A(a)\phi(a,t) - A(b)\phi(b,t) = \left.\frac{dU}{dt}\right|_{\text{rod}}.$$

Factor a minus sign from the left side.

$$-[A(b)\phi(b,t) - A(a)\phi(a,t)] = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

By the fundamental theorem of calculus, the term in square brackets is an integral.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\phi(x)]\,dx = \left.\frac{dU}{dt}\right|_{\text{rod}}$$

The thermal energy in the rod U is obtained by integrating the thermal energy density $e(x, t)$ over the rod's volume.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\phi(x)]\,dx = \frac{d}{dt}\int_{\text{rod}} e(x,t)\,dV$$

The volume differential is $dV = A(x)\,dx$.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\phi(x)]\,dx = \frac{d}{dt}\int_a^b e(x,t)A(x)\,dx$$

The thermal energy density is the mass density ρ times specific heat c times temperature $u(x,t)$.

$$-\int_a^b \frac{\partial}{\partial x}[A(x)\phi(x)]\,dx = \frac{d}{dt}\int_a^b \rho c u(x,t)A(x)\,dx$$

Bring the minus sign and derivative inside the integrals.

$$\int_a^b \left\{-\frac{\partial}{\partial x}[A(x)\phi(x)]\right\}dx = \int_a^b \rho c\frac{\partial u}{\partial t}A(x)\,dx$$

The integrands must be equal to one another.

$$-\frac{\partial}{\partial x}[A(x)\phi(x)] = \rho c\frac{\partial u}{\partial t}A(x)$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0\frac{\partial u}{\partial x},$$

where K_0 is a proportionality constant known as the thermal conductivity. As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial x}\left[-K_0A(x)\frac{\partial u}{\partial x}\right] = \rho c\frac{\partial u}{\partial t}A(x)$$

Therefore, the governing equation for the temperature is

$$\rho c\frac{\partial u}{\partial t} = \frac{K_0}{A(x)}\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right],$$

or, dividing both sides by ρc and setting $k = K_0/\rho c$,

$$\frac{\partial u}{\partial t} = \frac{k}{A(x)}\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right].$$